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K. Balasubramanian

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# **Group Representations and Multinomial Combinatorics of the Icosahedral Symmetry**

K. Balasubramanian\*

*Center for Image Processing and Integrated computing, University of California  
Davis, Livermore, CA 94550; University of California*

*Chemistry and Material Science Directorate*

*Lawrence Livermore National Laboratory*

*Livermore, California 94550;*

*and Glenn T Seaborg Center, Lawrence Berkeley Laboratory, University of  
California, Berkeley, CA 94720*

The icosahedral symmetry is one of the most intriguing symmetries, as it not only presents challenge but it appears in many fullerenes and high energetic materials such as the dodecahedral  $N_{20}$ . We have considered the combinatorics of all irreducible representations of the icosahedral symmetry for a number of multinomial partitions for vertex, face and edge colorings in this work. We have constructed the combinatorial tables for all irreducible representations for various multinomial partitions of colorings for the vertices, edge and faces of the icosahedron. These techniques should have important applications to enumerations and spectroscopy of fullerenes and high-energy materials such as  $N_{20}$ .

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\* Address correspondence at [kbala@ucdavis.edu](mailto:kbala@ucdavis.edu) or fax:925-422-6810

## **1. Introduction.**

The icosahedral symmetry [1-19] is quite interesting and intriguing as it appears in fullerenes including the celebrated Buckminsterfullrene,  $C_{60}$  [1-5], high-energy materials such as the dodecahedral  $N_{20}$  [7-12] and  $C_{60}$  analog,  $C_{48}N_{12}$  [13] and boron hydride species such as  $B_{12}H_{12}^{-2}$  [14], dodecahedral molecule  $C_{20}H_{20}$  [6], which has been experimentally isolated [6] etc. The carbon-based fullerenes and related high-energy materials containing nitrogen have changed the earlier belief that five-fold axes do not occur commonly in structures. The study of icosahedral symmetry and its properties have become thus quite important and interesting. There have been several studies that were focused on the applications of combinatorics and graph theory to structures and spectroscopy [14-40]. These studies have been concerned with enumeration of isomers, chirality, spectroscopy, nuclear spin statistics, etc. There have been generalizations of combinatorial techniques beyond the celebrated Pólya's theorem to all irreducible representations [20] and by the current author to generalization of De Bruijn's theorem to all irreducible representations [25].

The techniques that have been presented in the literature, especially for the icosahedral symmetry, have been restricted to two or at most 3 kinds of substituents or colors or ligands. The combinatorics of applying all irreducible representations for multiple types of colors or ligands becomes complex not only due to many different types of partitions for such colors but also because of the combinatorial explosion that occurs, as the number of different types of colors goes up. Thus computing these numbers become quite challenging. When these techniques are applied to cases with 2 colors they become binomial expansions, and are much simpler. But the techniques rapidly evolve into multinomial combinatorial methods with rapidly growing coefficients for multiple ligands and colors. Thus a full combinatorial classification of these irreducible patterns requires exhaustive types of all possible distinct colors. For example, the vertices of an icosahedron can be colored with 12 different kinds of colors, and the edges of an icosahedron up to 30 different colors, and so on. The combinatorial complexity grows exponentially with the color partition, and the associated Young's diagram becomes complex with multiple rows. It seems that it would be quite interesting to have exhaustive tables at least for the vertex colorings as a function of Young's diagram for all irreducible representations of the icosahedron. Such generating functions can be quite useful for a number of applications that involve the icosahedral symmetry such as enumeration, chirality, spin statistics, and so on. For example, the naturally occurring  $^{209}Bi$  has a

nuclear spin of 9/2, which yields 10 distinct orientations for the magnetic spin orientations, given by  $m_f = -9/2, -7/2, -5/2, -3/2, -1/2, 1/2, 3/2, 5/2, 7/2$ , and 9/2. If each of these orientations is mapped into a distinct type of color the nuclear spin statistics of bismuth clusters with icosahedral symmetry requires multinomial combinatorics with 10 terms.

The objective of this study is to consider multinomial combinatorics of the icosahedron. By the application of combinatorial methods with irreducible representations, we have constructed exhaustive combinatorial tables for all irreducible representations of the icosahedral vertex colorings represented by the Young's diagrams or partitions of 12 for all irreducible representations, selected edge colorings and face colorings of the icosahedron. As a particular case, we have also discussed the chirality of the associated colorings of vertices, faces and edges of the icosahedron.

## 2. Multinomial Combinatorics

The basic combinatorial techniques use character values of the irreducible representations of the  $I_h$  point group together with multinomial expansions. Multinomial expansions are constructed in terms of ordered partitions of  $n$ , denoted by,  $[n]$  into  $p$  parts (composition of the integer  $n$  into  $p$  parts) such that

$$n_1 \geq 0, n_2 \geq 0, \dots, n_p \geq 0, \quad \sum_{i=1}^p n_i = n.$$

A multinomial expansion in  $\lambda$ 's is defined as

$$(\lambda_1 + \lambda_2 + \dots + \lambda_p)^n = \sum_{[\lambda]} \begin{pmatrix} n \\ n_1 & n_2 & \dots & n_p \end{pmatrix} \lambda_1^{n_1} \lambda_2^{n_2} \dots \lambda_p^{n_p},$$

where  $\begin{pmatrix} n \\ n_1 & n_2 & \dots & n_p \end{pmatrix}$  are the multinomial coefficients and the sum is over all such ordered partitions. The multinomial coefficients are computed as

$$\begin{pmatrix} n \\ n_1 & n_2 & \dots & n_p \end{pmatrix} = \frac{n!}{n_1! n_2! \dots n_p!}$$

A useful result for computing the multinomial coefficients is

$$\binom{n+q}{n_1 \ n_2 \ \dots \ n_p} = \sum_{[k]=n} \binom{n}{k_1 \ k_2 \ \dots \ k_p} \binom{q}{n_1 - k_1 \ n_2 - k_2 \ \dots \ n_p - k_p},$$

where  $[k]$  stands for all ordered partitions of  $n$  such that  $k_1+k_2+\dots+k_p=n$ , with  $k_i$  non-negative integers.

The present author [15,16,25] has generalized Pólya's theorem [23,24] to all irreducible representations of a group. This was proposed earlier by Williamson [31] and Merris [32] with a single group action but the present author provided a physical interpretation and generalized the technique to multiple group actions. In this technique a polynomial is constructed using the character values for each irreducible representation of the group. Suppose a symmetry operator  $T_G^\chi$  is constructed as

$$T_G^\chi = \frac{1}{|G|} \sum_{g \in G} \chi(g) P(g)$$

where  $\chi(g)$  be the character value of  $g \in G$  for an irreducible representation  $\Gamma$  in the group  $G$ , and  $P(g)$  is a permutation operator for  $g$ . A weighted permutation operator is obtained by introducing a weight  $r$  for each kind of color in a set  $R$  with  $D$  being the set of vertices, edges or faces of the icosahedron. A coloring is simply a map  $f$  from the set  $D$  to the set  $R$ . The weight of a function  $f$  from  $D$  to  $R$  is given by

$$W(f) = \prod_{i=1}^n w(f(i))$$

We can define a permutational operator for each weight  $W$ , and denote it by  $P_w(g)$ . In the most general form of matrix representations of  $P_w(g)$ , the tensor version of generalization of Pólya's theorem for all irreducible representations in the group would simply use the above matrix operators. However, a more convenient character version is obtained by considering the trace of  $P_w(g)$  given by

$$Tr(P_w(g)) = \sum_f^{(g)} W(f),$$

where the sum is over all  $f$  for which  $gf = f$ , and consequently,

$$T_G^{W,\chi} = \frac{1}{|G|} \sum_{g \in G} \chi(g) P_w(g)$$

$$\text{Tr} T_G^{W,\chi} = \frac{1}{|G|} \sum_{g \in G} \chi(g) \text{Tr}[P_w(g)] = \frac{1}{|G|} \sum_{g \in G} \chi(g) \sum_f^{(g)} W(f)$$

The above expression is tantamount to replacement of every  $s_k$  in the generalized character cycle index (GCCI)  $P_G^\chi$  for every irreducible representation as defined below:

$$P_G^\chi = \frac{1}{|G|} \sum_{g \in G} \chi(g) s_1^{b_1} s_2^{b_2} \dots s_n^{b_n},$$

where the sum is over all elements of the group and  $s_1^{b_1} s_2^{b_2} \dots s_n^{b_n}$  is the cyclic polynomial representation if  $g$  in  $G$  generates  $b_1$  cycles of length 1,  $b_2$  cycles of length 2, ...,  $b_n$  cycles of length  $n$  when  $g$  acts on the set of elements  $D$ . The difference between Pólya's cycle index and the one above is that for each irreducible representation, we have a cycle index (although may not be unique) due to the character values multiplying the polynomial. We carry out a multinomial replacement of every  $s_k$  in the above expression by the following recipe. Suppose  $n$  is the number of elements in the set  $D$ , and  $R$  consists of  $n$  distinct types of colors, for example, white, green, yellow, purple, magenta, red, and so on. Let  $\lambda_1, \lambda_2, \dots, \lambda_n$  be  $n$  distinct weights for these colors. The multinomial function is computed by replacing every  $s_k$  in the above GCCI by  $\lambda_1^k + \lambda_2^k + \dots + \lambda_n^k$ . That is,

$$GF^\chi = P_G^\chi (s_k \rightarrow \sum_i \lambda_i^k)$$

The GF thus obtained above provides for generating functions for the various kinds of maps with different colorings such that each coloring transforms according to the irreducible representation  $\Gamma$  whose character is  $\chi$ . For example, for the identity representation the GF enumerates isomers or equivalence classes. If  $\Gamma$  is the anti-symmetric irreducible representation with  $-1$  character values for all improper rotations, the GF enumerates all chiral isomers. In general  $GF^\chi$  enumerates

equivalence classes of functions from D to R that transform according to the irreducible representation  $\Gamma$ .

As we showed above, the various terms in the GF have all ordered partitions of n into various parts given by Young's diagrams. Some of the terms in the complete generating function are combinatorially equivalent. To illustrate the terms  $\lambda_1^8 \lambda_2^2 \lambda_3 \lambda_4^1$ ,  $\lambda_1^8 \lambda_3^2 \lambda_4 \lambda_6^1$ ,  $\lambda_2^8 \lambda_4^2 \lambda_6 \lambda_7^1$ , etc., are all equivalent as their coefficients in the multinomial expansion would be the same. The unique terms of the multinomial GF's are the Young diagrams of partition n into various parts.

### 3. Application to the Icosahedron.

The icosahedral point group  $I_h$  is probably the most complex among point groups exhibiting multiple axes of rotations, planes and improper axes of rotations. We start with the vertex colorings of the icosahedron with 12 different colors. In this case D is the set of 12 vertices of the icosahedron and R is a set of 12 distinct colors such as white, yellow, blue, green, red, purple, magenta, cyan, and so on. It is easy to show that there are  $12^{12}$  possible vertex colorings of the icosahedron, which when symmetry-adapted would transform as various irreducible representations of the  $I_h$  group. The number of totally symmetric representations is also the number of equivalence classes or positional isomers for the various vertex colorings.

Table 1 shows the GCCIs for vertex, face and edge colorings of the icosahedron. The character values are the same for all three cases; the main difference is the permutation representations for each conjugacy class depending on whether the set D is the set of vertices, edges or faces. For example, the cycle index polynomial for the  $A_u$  irreducible representation of the  $I_h$  group for the vertex colorings is given by

$$P_{I_h}^{A_u} = \frac{1}{120} [s_1^{12} + 24s_1^2 s_5^2 + 20s_3^4 + (15-1)s_2^6 - 24s_2^1 s_{10} - 20s_6^2 - 15s_1^4 s_2^4]$$

If we replace every  $s_k$  by  $\sum_{i=1}^{12} \lambda_i^k$  in the above expression, we obtain,

$$P_{I_h}^{A_u} = \frac{1}{120} \left[ (\lambda_1 + \lambda_2 + \dots + \lambda_{12})^{12} + 24(\lambda_1 + \lambda_2 + \dots + \lambda_{12})^2 (\lambda_1^5 + \lambda_2^5 + \dots + \lambda_{12}^5)^2 + 20(\lambda_1^3 + \lambda_2^3 + \dots + \lambda_{12}^3)^4 + 14(\lambda_1^2 + \lambda_2^2 + \dots + \lambda_{12}^2)^6 - 24(\lambda_1^2 + \lambda_2^2 + \dots + \lambda_{12}^2)(\lambda_1^{10} + \lambda_2^{10} + \dots + \lambda_{12}^{10}) - 20(\lambda_1^6 + \lambda_2^6 + \dots + \lambda_{12}^6)^2 - 15(\lambda_1 + \lambda_2 + \dots + \lambda_{12})^4 (\lambda_1^2 + \lambda_2^2 + \dots + \lambda_{12}^2)^4 \right]$$

The above expression can be simplified by collecting the coefficient of a typical term

$\lambda_1^{b_1} \lambda_2^{b_2} \dots \lambda_{12}^{b_{12}}$  which enumerates number of times the irreducible representation  $A_u$  occurs in the set of functions that contain  $b_1$  colors of type 1,  $b_2$  colors of type 2,..... $b_{12}$  colors of the type 12. Since the  $A_u$  representation has all  $-1$  character values for the improper rotations, it corresponds to the number of chiral vertex colorings of the icosahedron with the above distribution of colors.

Table 2 shows the full combinatorial results for the vertex colorings of the icosahedron for all irreducible representations and all partitions of 12. In table 2, we show only the unique terms as determined by the Young diagrams for 12. There is more than one term for each young diagram,

for example, the term  $12 \ 0 \ 0 \ \dots \ 0$  could represent  $\lambda_1^{12} \text{ or } \lambda_2^{12} \dots \lambda_{12}^{12}$ . Thus each Young diagram represents multiple ordered partitions, and only a unique coefficient is shown in Table 2. As can be seen from table 2, the first non-zero binomial coefficient occurs for the  $A_u$  representation for the partition [8+4], which has two chiral colorings. This would correspond to the term  $\lambda_1^8 \lambda_2^4$ . The first nonzero trinomial coefficient occurs for the partition [9+2+1] with 2 chiral colorings. The  $T_{1g}$  representation has the first non-zero binomial term as [9+3] and the first non-zero trinomial term as [10+1+1]. We can verify the correctness of all numbers using the fact that the sum of all numbers for each irreducible representation multiplied by the number of times that partition occurs for various irreducible representations in Table 2 should be equal to

$$GF^\chi = P_G^\chi (s_k \rightarrow 12)$$

For example, the above result for the  $A_u$  representation gives

$$P_{I_h}^{A_u} = \frac{1}{120} [(12^{12} + 24 \times 12^2 \times 12^2 + 20 \times 12^4 + 14 \times 12^6 - 24 \times 12^2 - 20 \times 12^2 - 15 \times 12^8)] \\ = 8320953630$$

Table 3 shows all such frequencies obtained directly by the above technique for all of the irreducible representations of the  $I_h$  point group. It can also been that due to the fact that only the first term in the GF contributes to certain terms of the multinomial, and thus they can be readily obtained. For example, this appears to be the case for the last few rows for the vertex colorings according to the irreducible representation  $\Gamma$ . The last row is always given by

$$\frac{1}{120} [12 \times \dim(\Gamma)]$$

The one previous to the last row is given by

$$\frac{1}{720} [12 \times \dim(\Gamma)]$$

The last but one row is given by

$$\frac{1}{480} [12 \times \dim(\Gamma)]$$

Similar results can be obtained for a few other rows in Table 2, which have contributions only from the leading term.

Table 4 shows the face colorings of the icosahedron with up to 4 kinds of colors. We have restricted our table to 4 colors, as too many terms are generated for cases beyond 4 types of colors. As seen from Table 4, the first  $A_u$  non-zero representation appears for the partition 18+2 then 18+1+1 both of which suggest that to produce a chiral structure out of an icosahedron by coloring the faces, one needs at least 18 colors of one kind and 2 colors of another kind for the binomial and 18,1,1 colorings for the trinomial. Analogous to the vertex colorings, the face colorings for the partition  $1^{20}$  (not shown in table 4) is given by

$$\frac{1}{120} [20 \times \dim(\Gamma)].$$

Tables 5 and 6 show our edge colorings of the icosahedron for all the  $g$  and  $u$  irreducible representations, respectively for up to 4 types of colors as a function of partition of 30 as there are 30 edges for the icosahedron. The first non-zero binomial coloring occurs for the 28+2 partition for the chiral representation  $A_u$  with 3 colorings. The corresponding trinomial term occurs for 28+1+1 with 6 chiral colorings. The corresponding tetranomial term with non-zero chiral coloring is

attributed to the term 27+1+1+1 with 200 chiral colorings. As seen from Tables 5 and 6, the coefficients grow rapidly as a function of the complexity of the Young diagram, that is, as the number of rows of the diagram grows. We have used the quadruple precision arithmetic to compute these coefficients. Even though we do not show all of the results in tables 5 and 6, the coefficient of the term with the  $1^{30}$  partition with 30 rows for the Young diagram is given by

$$\frac{1}{120} [30! \times \dim(\Gamma)]$$

The generalized character cycle indices that we used to enumerate various colorings can also be used in other applications. For example, instead of the usual Pólya's substitution we can substitute in the icosahedral cycle indices other symmetric functions known as the S-functions [41]. For example, one may replace every  $s_k$  by

$$\lambda_1^k \lambda_2^k + \lambda_1^k \lambda_3^k + \dots + \lambda_1^k \lambda_n^k + \lambda_2^k \lambda_3^k + \dots + \lambda_2^k \lambda_3^k + \lambda_{n-1}^k \lambda_n^k = \sum \lambda_i^k \lambda_j^k$$

in the generalized cycle indices to produce interesting new combinatorial identities for each irreducible representation. We may also consider the replacement by tri-symmetric functions of the kind  $\sum \lambda_1^k \lambda_2^k \lambda_3^k$  where the sum involves all possible symmetric combinations with three terms at a time. It can be seen that the method could be generalized to a multinomial S-function.

The enumeration of electronic configurations can be achieved [30] by choosing weights 1,  $w, w^2, \dots, w^n$ . If we stop with weights 1,  $w$ , and  $w^2$  this corresponds to filling the orbitals with zero, 1 or 2 electrons as by Pauli exclusion principle only as at most 2 electrons are allowed in any orbital. For example, the GF for the  $A_g$  representation of the vertex group of icosahedron for the general weights is given by

$$P_{I_h}^{A_g} = \frac{1}{120} \left[ (1 + w + w^2 + w^3 + \dots + w^{12})^{12} + 24(1 + w + w^2 + w^3 + \dots + w^{12})^2 (1 + w^5 + w^{10} + w^{15} + \dots + w^{60})^2 + 20(1 + w^3 + w^6 + w^9 + \dots + w^{36})^6 + 16(1 + w^2 + w^4 + w^8 + \dots + w^{24})^6 - 24(1 + w^2 + w^4 + w^8 + \dots + w^{24}) (1 + w^{10} + w^{20} + w^{30} + \dots + w^{120}) - 20(1 + w^6 + w^{12} + w^{18} + \dots + w^{72})^2 - 15(1 + w + w^2 + w^3 + \dots + w^{12})^{12} \times (1 + w^2 + w^4 + w^8 + \dots + w^{24})^4 \right]$$

Other variations that lead to infinite power series in multinomial weights are also feasible. There is much to be desired in such powerful multinomial combinatorial theory.

A large number of applications of multinomial combinatorics of the kind developed here is possible. One of the important applications would be to nuclear spin statistics of molecular rovibronic levels [14-16,20]. Clusters of many species such as bismuth clusters exhibit high spin states thus spanning a multiple set of nuclear spin projections. Each projection becomes a variable in the multinomial combinatorial expansion. For example, nuclear spin properties of bismuth clusters would require such elaborate multivariable combinatorial expansions. Many inorganic species exhibit multiple ligands. If each type of ligand were to be associated with a type of color then enumeration of possible ligand substitutions for the icosahedron would become a multinomial problem with ligand partitions represented by the various Young diagrams.

Longuet-Higgins [42] has formulated the symmetry groups of non-rigid molecules as permutation-inversion groups. The rovibronic levels and nuclear spin statistical weights of non-rigid molecules and clusters would require multinomial combinatorial representations if the nuclei in the cluster have high-spin characteristics. The rovibronic levels of such species are split into tunneling levels as the non-rigid cluster tunnels through surmountable potential minima. The nuclear spin statistical weights of the tunneling levels would require multinomial polynomial expansions in terms of the character representations of the permutation-inversion groups of non-rigid molecules. Thus many applications seem to be feasible to molecular physics.

#### 4. Conclusion

In this work we have obtained the full combinatorial tables for all irreducible representations of the icosahedral group for the vertex, edge and face colorings of the icosahedron using multinomial combinatorics. The tables provided full enumerations for all possible vertex colorings that transform according to the various irreducible representations of the  $I_h$  group in terms of partitions or the young diagram. The results also included the chiral edge, vertex, and face colorings for all young diagrams. The results for other irreducible representations were interpreted. We have shown that there is rich combinatorics in the GCCIs when replacements are made for each term by a multinomial of S-functions.

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## 6. References

- [1] Kroto, H. W., Heath, J. R., O'Brien, S.C., Curl, R. F., and Smalley, R. E., 1985, *Nature*, **318**, 162.
- [2] Curl, R. F. and Smalley, R. E., 1988, *Science*, **242**, 1017.
- [3] Kroto, H. W., 1989, *Comput Math. Appl.* **17**, 417.
- [4] Kroto, H.W., Allaf, A. W., and Blam, S. P., 1989, *Chem. Rev.* **91**, 1213.
- [5] Chibante, L. P. F., and Smalley, R. E., 1992, *Complete Buckminsterfullerene Bibliography*.
- [6] Paquette, L. A., 1989, *Chem. Rev.*, **89**, 1051.
- [7] Bliznyuk, A. A., Shen, M., and Schaefer III, H. F., 1992, *Chem. Phys. Letters* **198**, 249.
- [8] Seidl, E. T., and Schaefer III, H. F., 1988, *J. Chem. Phys.* **88**, 7043.
- [9] Balasubramanian, K., 1993, *Chem. Phys. Lett.* **202**, 272.
- [10] Alkorta, I., Elguero, J., Rozas, I., and Balaban, A. T., 1991, *J. Mol. Struct.* **228**, 47.
- [11] Schulman, J. M. and Disch, R.J., 1978, *J. Am. Chem. Soc.*, **100**, 5677.
- [12] Chen, C., Lu, L.-H., and Wang, Y.-W., 1992, *J. Mol. Struc.*, **253**, 1.
- [13] Manaa, M. R.; Sprehn, D. W.; Ichord, H. A., 2002, *J. Am. Chem. Soc.* **124**, 13990.
- [14] Balasubramanian, K., Strauss, H., and Pitzer, K. S., 1982, *J. Mol. Spectrosc.*
- [15] Balasubramanian, K., 1991, *Chem. Phys. Lett.* **183**, 292.
- [16] Balasubramanian, K., 1992, *Chem. Phys. Lett.* **200**, 649.
- [17] Colpa, J. P., and Temme, F., 1991, *Z. Phys. D.* **23**, 187.
- [18] Harter, W. G., and Weeks, D. E., 1989, *J. Chem. Phys.* **90**, 4727
- [19] Harter, W. G., and Reimer, T. C., 1992, *J. Chem. Phys.* **194**, 230; **198**, 429E.
- [20] Balasubramanian, K., 1985, *Chemical Rev.* **85**, 599.
- [21] Balasubramanian, K., 1979, *Theor. Chim. Acta*, **51**, 37
- [22] Balaban, A. T., 1976, “*Chemical Applications of Graph Theory*”, Academic, New York, NY
- [23] Pólya, G., and Read, R.C., 1987, “*Combinatorial Enumeration of Groups, Graphs, and Chemical Compounds*”, Springer, New York and Berlin.

- [24] Pólya, G., 1937, *Acta. Math.* **68**, 145.
- [25] Balasubramanian, K., 1993, *J. Math Chem.* **14**, 113
- [26] Robinson, R. W., Harary, F. and Balaban, A. T., 1976, *Tetrahedron* **32**, 355.
- [27] Rouvray, D. H., 1974, *Chem. Soc. Rev.* **3**, 355.
- [28] Harary, F. and Palmer, E. M., 1973, "Graphical Enumeration", Academic press, New York, NY
- [29] Balasubramanian, K., 1981, *J. Chem. Phys.* **74**, 6824
- [30] Balasubramanian, K., 1981, *Int. J. Quant. Chem.* **20**, 1255
- [31] Williamson, S. G., 1971, *J. Comb. Theory*, **11**, 122
- [32] Merris, R., 1980, *Linear Algebra and Applications*, **29**, 255.
- [33] King, R. B. 1987 *J. Math. Chem.* **1**, 15.
- [34] King R. B., 1987, *J. Math. Chem.* **1**, 55.
- [35] King, R. B., 2003, in *Chemical Explanation: Characteristics, Development, Autonomy*, ed. Earley, J. E., *Ann. N. Y. Acad. Sci.*, **988**, 158
- [36] Balasubramanian, K., 1991, *Chemical Physics Letters*, **183**, 292
- [37] Balasubramanian, K., 1991, *J. Chem. Phys.* **95**, 8273.
- [38] Balasubramanian, K., 1993, *Chemical Physics Lett.* **210**, 153.
- [39] Balasubramanian, K., 1992, *Chemical Physics Lett.* **200**, 649.
- [40] Balasubramanian, K., 1996, *Chem. Phys. Lett.* **260**, 476
- [41] Littlewood, D. E., 1940, "Theory of Group Representation and Matrix Product of Groups", Clarendon, Oxford.
- [42] Longuet-Higgins, H.C., 1963, *Molecular Physics* **6**, 445.

Table 1 GCCI table for the vertex/face/edge colorings of the icosahedron

Vertex	$1^{12}$	$1^2 5^2$	$3^4$	$2^6$	$2^6$	$2 \ 10$	$6^2$	$1^4 2^4$
Face	$1^{20}$	$5^4$	$1^2 3^6$	$2^{10}$	$2^{10}$	$10^2$	$2 \ 6^3$	$1^4 2^8$
Edge	$1^{30}$	$5^6$	$3^{10}$	$1^2 2^{14}$	$2^{15}$	$10^3$	$6^5$	$1^4 2^{13}$
Order	1	24	20	15	1	24	20	15
$A_g$	1	1	1	1	1	1	1	1
$A_u$	1	1	1	1	-1	-1	-1	-1
$T_{1g}=T_{2g}$	3	1/2	0	-1	3	1/2	0	-1
$T_{1u}=T_{2u}$	3	1/2	0	-1	-3	-1/2	0	1
$G_g$	4	-1	1	0	4	-1	1	0
$G_u$	4	-1	1	0	-4	1	-1	0
$H_g$	5	0	-1	1	5	0	-1	1
$H_u$	5	0	-1	1	-5	0	1	-1

Table 2 Icosahedral vertex combinatorics for all irreducible representations up to 12 colors

Partition	$A_g$	$T_{1g}=T_{2g}$	$G_g$	$H_g$	$A_u$	$T_{1u}=T_{2u}$	$G_u$	$H_u$
12 0 0 0 0 0 0 0	1	0	0	0	0	0	0	0
11 1 0 0 0 0 0 0	1	0	0	1	0	1	0	0
10 2 0 0 0 0 0 0	3	0	2	5	0	2	2	2
9 3 0 0 0 0 0 0	5	3	8	11	0	8	8	6
8 4 0 0 0 0 0 0	10	7	17	27	2	14	16	18
7 5 0 0 0 0 0 0	12	15	26	38	2	25	26	28
6 6 0 0 0 0 0 0	18	16	32	46	6	26	30	34
10 1 1 0 0 0 0 0	3	2	4	7	0	5	4	4
9 2 1 0 0 0 0 0	9	13	22	31	2	20	22	24
8 3 1 0 0 0 0 0	23	43	66	89	10	56	66	76
7 4 1 0 0 0 0 0	42	90	132	174	24	108	132	156
6 5 1 0 0 0 0 0	58	128	184	242	36	150	184	220
8 2 2 0 0 0 0 0	37	63	100	137	20	78	98	118
7 3 2 0 0 0 0 0	80	184	264	344	52	212	264	316
6 4 2 0 0 0 0 0	142	322	464	606	104	356	460	564
5 5 2 0 0 0 0 0	160	395	554	714	118	437	554	672
6 3 3 0 0 0 0 0	178	440	618	790	134	484	618	746
5 4 3 0 0 0 0 0	258	666	924	1182	204	720	924	1128
4 4 4 0 0 0 0 0	330	828	1158	1488	270	882	1152	1422
9 1 1 1 0 0 0 0	14	30	44	58	8	36	44	52
8 2 1 1 0 0 0 0	57	141	198	255	42	156	198	240
7 3 1 1 0 0 0 0	144	384	528	672	120	408	528	648
6 4 1 1 0 0 0 0	246	678	924	1170	216	708	924	1140
5 5 1 1 0 0 0 0	296	814	1108	1404	260	850	1108	1368
7 2 2 1 0 0 0 0	216	576	792	1008	180	612	792	972
6 3 2 1 0 0 0 0	488	1360	1848	2336	436	1412	1848	2284
5 4 2 1 0 0 0 0	726	2046	2772	3498	660	2112	2772	3432
5 3 3 1 0 0 0 0	960	2736	3696	4656	888	2808	3696	4584
4 4 3 1 0 0 0 0	1194	3426	4620	5814	1116	3504	4620	5736
6 2 2 2 0 0 0 0	748	2028	2776	3524	668	2100	2768	3436
5 3 2 2 0 0 0 0	1440	4104	5544	6984	1332	4212	5544	6876
4 4 2 2 0 0 0 0	1818	5118	6936	8754	1692	5232	6924	8616
4 3 3 2 0 0 0 0	2376	6864	9240	11616	2244	6996	9240	11484
3 3 3 3 0 0 0 0	3156	9168	12324	15468	3012	9312	12324	15324
8 1 1 1 1 0 0 0	102	294	396	498	96	300	396	492
7 2 1 1 1 0 0 0	408	1176	1584	1992	384	1200	1584	1968
6 3 1 1 1 0 0 0	936	2760	3696	4632	912	2784	3696	4608
5 4 1 1 1 0 0 0	1404	4140	5544	6948	1368	4176	5544	6912
6 2 2 1 1 0 0 0	1416	4128	5544	6960	1356	4188	5544	6900
5 3 2 1 1 0 0 0	2808	8280	11088	13896	2736	8352	11088	13824
4 4 2 1 1 0 0 0	3510	10350	13860	17370	3420	10440	13860	17280
4 3 3 1 1 0 0 0	4656	13824	18480	23136	4584	13896	18480	23064
5 2 2 2 1 0 0 0	4224	12408	16632	20856	4092	12540	16632	20724
4 3 2 2 1 0 0 0	7008	20712	27720	34728	6852	20868	27720	34572
3 3 3 2 1 0 0 0	9312	27648	36960	46272	9168	27792	36960	46128
4 2 2 2 2 0 0 0	10572	31020	41592	52164	10308	31260	41568	51876
3 3 2 2 2 0 0 0	13992	41448	55440	69432	13728	41712	55440	69168
7 1 1 1 1 1 0 0	792	2376	3168	3960	792	2376	3168	3960

6 2 1 1 1 1 0 0	2784	8304	11088	13872	2760	8328	11088	13848
5 3 1 1 1 1 0 0	5544	16632	22176	27720	5544	16632	22176	27720
4 4 1 1 1 1 0 0	6948	20772	27720	34668	6912	20808	27720	34632
5 2 2 1 1 1 0 0	8352	24912	33264	41616	8280	24984	33264	41544
4 3 2 1 1 1 0 0	13896	41544	55440	69336	13824	41616	55440	69264
3 3 3 1 1 1 0 0	18480	55440	73920	92400	18480	55440	73920	92400
4 2 2 2 1 1 0 0	20880	62280	83160	104040	20700	62460	83160	103860
3 3 2 2 1 1 0 0	27792	83088	110880	138672	27648	83232	110880	138528
3 2 2 2 2 1 0 0	41736	124584	166320	208056	41424	124896	166320	207744
2 2 2 2 2 2 0 0	62736	186768	249504	312240	62184	187272	249456	311640
6 1 1 1 1 1 1 0	5544	16632	22176	27720	5544	16632	22176	27720
5 2 1 1 1 1 1 0	16632	49896	66528	83160	16632	49896	66528	83160
4 3 1 1 1 1 1 0	27720	83160	110880	138600	27720	83160	110880	138600
4 2 2 1 1 1 1 0	41616	124704	166320	207936	41544	124776	166320	207864
3 3 2 1 1 1 1 0	55440	166320	221760	277200	55440	166320	221760	277200
3 2 2 2 1 1 1 0	83232	249408	332640	415872	83088	249552	332640	415728
2 2 2 2 2 1 1 0	124920	374040	498960	623880	124560	374400	498960	623520
5 1 1 1 1 1 1 1	33264	99792	133056	166320	33264	99792	133056	166320
4 2 1 1 1 1 1 1	83160	249480	332640	415800	83160	249480	332640	415800
3 3 1 1 1 1 1 1	110880	332640	443520	554400	110880	332640	443520	554400
3 2 2 1 1 1 1 1	166320	498960	665280	831600	166320	498960	665280	831600
2 2 2 2 1 1 1 1	249552	748368	997920	1247472	249408	748512	997920	1247328
4 1 1 1 1 1 1 1	166320	498960	665280	831600	166320	498960	665280	831600
3 2 1 1 1 1 1 1	332640	997920	1330560	1663200	332640	997920	1330560	1663200
2 2 2 1 1 1 1 1	498960	1496880	1995840	2494800	498960	1496880	1995840	2494800
2 2 1 1 1 1 1 1 1	997920	2993760	3991680	4989600	997920	2993760	3991680	4989600
3 1 1 1 1 1 1 1 1	665280	1995840	2661120	3326400	665280	1995840	2661120	3326400
1 1 1 1 1 1 1 1 1	3991680	11975040	15966720	19958400	3991680	11975040	15966720	19958400

Table 3 Frequencies of all irreducible representations in vertex edge combinatorics of icosahedron for all 12 colors.

$A_g$	8345970370
$A_u$	8320953630
$T_{1g}=T_{2g}$	24987401010
$G_g$	33333366330
$H_g$	41679331650
$T_{1u}=T_{2u}$	25012350990
$G_u$	33333299670
$H_u$	41654248350

<sup>a</sup> Sum of frequencies x dimension of the reps for all irreducible reps is verified to be  $12^{12}$ .

Table 4 Icosahedral Face Combinatorics for all irreducible representations up to 4 colors

20	0	0	0	1	0	0	0	0	0	0
19	1	0	0	1	0	1	0	1	1	0
18	2	0	0	5	2	7	11	1	5	6
17	3	0	0	15	24	39	51	6	33	39
16	4	0	0	58	107	165	217	38	124	162
15	5	0	0	149	370	517	663	113	406	517
14	6	0	0	371	928	1299	1661	310	980	1290
13	7	0	0	693	1896	2589	3267	609	1980	2589
12	8	0	0	1135	3074	4209	5335	1022	3172	4194
11	9	0	0	1466	4136	5602	7058	1340	4262	5602
10	1	0	0	1648	4528	6172	7800	1510	4648	6156
18	1	1	0	5	8	13	17	2	11	13
17	2	1	0	34	80	114	148	23	91	114
16	3	1	0	176	472	648	818	151	497	648
15	4	1	0	674	1912	2586	3254	622	1964	2586
14	5	1	0	1984	5768	7752	9736	1892	5860	7752
13	6	1	0	4597	13496	18093	22675	4457	13636	18093
12	7	1	0	8501	25096	33597	42083	8305	25292	33597
11	8	1	0	12716	37672	50388	63104	12478	37910	50388
10	9	1	0	15536	46056	61592	77108	15270	46322	61592
16	2	2	0	274	698	972	1246	233	733	966
15	3	2	0	1337	3832	5169	6503	1249	3920	5169
14	4	2	0	4984	14408	19392	24376	4796	14572	19368
13	5	2	0	13720	40544	54264	67984	13412	40852	54264
12	6	2	0	29739	87864	117603	147333	29262	88284	117546
11	7	2	0	50696	150856	201552	252248	50080	151472	201552
10	8	2	0	69812	207364	277176	346988	69070	208022	277092
9	9	2	0	77370	230560	307930	385290	76600	231330	307930
14	3	3	0	6557	19288	25845	32387	6373	19472	25845
13	4	3	0	22802	67648	90450	113222	22438	68012	90450
12	5	3	0	59085	176064	235149	294219	58497	176652	235149
11	6	3	0	118002	352296	470298	588270	117162	353136	470298
10	7	3	0	185308	553736	739044	924292	184244	554800	739044
9	8	3	0	231550	692240	923790	1155310	230360	693430	923790
12	4	4	0	74014	219968	293982	367966	73286	220612	293898
11	5	4	0	176904	528528	705432	882336	175812	529620	705432
10	6	4	0	324428	968968	1293396	1617764	322888	970340	1293228
9	7	4	0	462820	1384760	1847580	2310340	461000	1386580	1847580
8	8	4	0	521000	1557610	2078610	2599610	519040	1559360	2078400
10	5	5	0	388788	1163166	1551948	1940736	387192	1164762	1551948
9	6	5	0	647706	1938888	2586594	3234270	645606	1940988	2586594
8	7	5	0	832592	2493016	3325608	4158200	830212	2495396	3325608
8	6	6	0	971840	2908192	3880032	4851824	969178	2910572	3879750
7	7	6	0	1109966	3324208	4434174	5544050	1107166	3327008	4434174
17	1	1	1	60	168	228	288	54	174	228
16	2	1	1	498	1440	1938	2436	471	1467	1938
15	3	1	1	2610	7728	10338	12942	2562	7776	10338
14	4	1	1	9744	29016	38760	48504	9636	29124	38760
13	5	1	1	27216	81312	108528	135744	27048	81480	108528
12	6	1	1	58917	176232	235149	294051	58665	176484	235149
11	7	1	1	100944	302160	403104	504048	100608	302496	403104
10	8	1	1	138756	415512	554268	693024	138378	415890	554268
9	9	1	1	154180	461680	615860	770020	153760	462100	615860

15	2	2	1	3928	11576	15504	19432	3824	11680	15504	19328
14	3	2	1	19480	58040	77520	97000	19280	58240	77520	96800
13	4	2	1	68040	203280	271320	339360	67620	203700	271320	338940
12	5	2	1	176680	528752	705432	882112	176036	529396	705432	881468
11	6	2	1	353192	1057672	1410864	1764056	352240	1058624	1410864	1763104
10	7	2	1	554856	1662216	2217072	2771928	553680	1663392	2217072	2770752
9	8	2	1	693500	2077840	2771340	3464840	692170	2079170	2771340	3463510
13	3	3	1	90618	271152	361770	452358	90282	271488	361770	452022
12	4	3	1	294290	881440	1175730	1469990	293590	882140	1175730	1469290
11	5	3	1	705936	2115792	2821728	3527664	704928	2116800	2821728	3526656
10	6	3	1	1294012	3879176	5173188	6467140	1292612	3880576	5173188	6465740
9	7	3	1	1848420	5541840	7390260	9238620	1846740	5543520	7390260	9236940
8	8	3	1	2079380	6234640	8314020	10393400	2077630	6236390	8314020	10391650
11	4	4	1	882504	2644656	3527160	4409664	881076	2646084	3527160	4408236
10	5	4	1	1940904	5818848	7759752	9700656	1938972	5820780	7759752	9698724
9	6	4	1	3234580	9698360	12932940	16167460	3231920	9701020	12932940	16164800
8	7	4	1	4158480	12469560	16628040	20786520	4155540	12472500	16628040	20783580
9	5	5	1	3881136	11638368	15519504	19400640	3878616	11640888	15519504	19398120
8	6	5	1	5821424	17457832	23279256	29100680	5818204	17461052	23279256	29097460
7	7	5	1	6652896	19951968	26604864	33257760	6649536	19955328	26604864	33254400
7	6	6	1	7761742	23277296	31039038	38800690	7757822	23281216	31039038	38796770
14	2	2	2	29352	86952	116304	145656	28968	87288	116256	145224
13	3	2	2	136024	406616	542640	678664	135296	407344	542640	677936
12	4	2	2	441952	1321712	1763664	2205616	440468	1323028	1763496	2203964
11	5	2	2	1059240	3173352	4232592	5291832	1057056	3175536	4232592	5289648
10	6	2	2	1942136	5817784	7759920	9702056	1939000	5820584	7759584	9698584
9	7	2	2	2773160	8312200	11085360	13858520	2769520	8315840	11085360	13854880
8	8	2	2	3120540	9350700	12471240	15591780	3116550	9354270	12470820	15587370
12	3	3	2	588509	1762936	2351445	2939939	587221	1764224	2351445	2938651
11	4	3	2	1764840	5289480	7054320	8819160	1762320	5292000	7054320	8816640
10	5	3	2	3881640	11637864	15519504	19401144	3878112	11641392	15519504	19397616
9	6	3	2	6468850	19397000	25865850	32334670	6464090	19401760	25865850	32329910
8	7	3	2	8316680	24939400	33256080	41572760	8311360	24944720	33256080	41567440
10	4	4	2	4853184	14546448	19399632	24252816	4848396	14550732	19399128	24247524
9	5	4	2	9702840	29095920	38798760	48501600	9696540	29102220	38798760	48495300
8	6	4	2	14555240	43643320	58198560	72753800	14546980	43650740	58197720	72744700
7	7	4	2	16632240	49879920	66512160	83144400	16623840	49888320	66512160	83136000
8	5	5	2	17463432	52374336	69837768	87301200	17455452	52382316	69837768	87293220
7	6	5	2	23284016	69833008	93117024	116401040	23274496	69842528	93117024	116391520
6	6	6	2	27166848	81470256	108637104	135803904	27155646	81480336	108635982	135791586
11	3	3	3	2352468	7053312	9405780	11758188	2350452	7055328	9405780	11756172
10	4	3	3	6468432	19397448	25865880	32334192	6464568	19401312	25865880	32330328
9	5	3	3	12935460	38796240	51731700	64667100	12930420	38801280	51731700	64662060
8	6	3	3	19402630	58194920	77597550	97000090	19396190	58201360	77597550	96993650
7	7	3	3	22174140	66508800	88682940	110856900	22167420	66515520	88682940	110850180
9	4	4	3	16169760	48494880	64664640	80834280	16162620	48502020	64664640	80827140
8	5	4	3	29103480	87292800	116396280	145499760	29094660	87301620	116396280	145490940
7	6	4	3	38804140	116390960	155195100	193999060	38793500	116401600	155195100	193988420
7	5	5	3	46563552	139670496	186234048	232797600	46553472	139680576	186234048	232787520
6	6	5	3	54324174	162948912	217273086	271597170	54312414	162960672	217273086	271585410
8	4	4	4	36382500	109113480	145495980	181878480	36369900	109124820	145494720	181864620
7	5	4	4	58205280	174587280	232792560	290997840	58191000	174601560	232792560	290983560
6	6	4	4	67909580	203682640	271592220	339501620	67892500	203698040	271590540	339482860
6	5	5	4	81485376	244424208	325909584	407394960	81469416	244440168	325909584	407379000
5	5	5	5	97780440	293311068	391091496	488871936	97765320	293326188	391091496	488856816

Table 5 Icosahedral Edge Combinatorics for g irreducible representations up to 4 colors.

Partition	$A_g$	$T_{1g}$	$G_g$	$H_g$
30 0 0 0	1	0	0	0
29 1 0 0	1	0	1	2
28 2 0 0	8	7	15	23
27 3 0 0	46	91	137	178
26 4 0 0	262	655	917	1179
25 5 0 0	1257	3495	4749	6006
24 6 0 0	5113	14703	19816	24904
23 7 0 0	17238	50622	67860	85098
22 8 0 0	49270	145873	195143	244413
21 9 0 0	119997	356928	476925	596862
20 10 0 0	251512	750094	1001597	1253109
19 11 0 0	456729	1364181	1820910	2277639
18 12 0 0	722750	2160561	2883311	3605951
17 13 0 0	1000251	2991744	3991995	4992246
16 14 0 0	1214376	3633261	4847637	6062013
15 15 0 0	1295266	3875366	5170622	6465762
28 1 1 0	9	20	29	38
27 2 1 0	113	293	406	519
26 3 1 0	937	2717	3654	4591
25 4 1 0	6019	17732	23751	29770
24 5 1 0	29835	88920	118755	148590
23 6 1 0	119106	355914	475020	594126
22 7 1 0	390754	1170026	1560780	1951534
21 8 1 0	1074073	3218072	4292145	5366218
20 9 1 0	2505217	7509788	10015005	12520222
19 10 1 0	5009719	15020291	20030010	25039729
18 11 1 0	8652111	25945179	34597290	43249401
17 12 1 0	12977523	38918412	51895935	64873458
16 13 1 0	16969947	50893968	67863915	84833862
15 14 1 0	19393980	58164780	77558760	96952740
26 2 2 0	1438	4050	5488	6926
25 3 2 0	12025	35477	47502	59527
24 4 2 0	74698	222235	296933	371631
23 5 2 0	357162	1067898	1425060	1782222
22 6 2 0	1367704	4095208	5462912	6830616
21 7 2 0	4295434	12873146	17168580	21464014
20 8 2 0	11272690	33795333	45068023	56340713
19 9 2 0	25045735	75104315	100150050	125195785
18 10 2 0	47583250	142702846	190286096	237869346
17 11 2 0	77858703	233516907	311375610	389234313
16 12 2 0	110297148	330819801	441116949	551414097
15 13 2 0	135747564	407163756	542911320	678658884
14 14 2 0	145443696	436248720	581692416	727136112
24 3 3 0	99270	296595	395865	495090
23 4 3 0	594750	1780350	2375100	2969850
22 5 3 0	2733042	8192418	10925460	13658502
21 6 3 0	10018926	30041154	40060080	50078826
20 7 3 0	30050878	90129182	120180060	150230938

19	8	3	0	75122905	225327245	300450150	375573055
18	9	3	0	158584995	475698795	634283790	792868365
17	10	3	0	285447591	856262979	1141710570	1427158161
16	11	3	0	441139257	1323322533	1764461790	2205601047
15	12	3	0	588182454	1764433476	2352615930	2940797754
14	13	3	0	678669180	2035887420	2714556600	3393225780
22	4	4	0	3417258	10239840	13657098	17074356
21	5	4	0	15027870	45062160	60090030	75117900
20	6	4	0	52590538	157725568	210316106	262906644
19	7	4	0	150242950	450657350	600900300	751143250
18	8	4	0	356815030	1070325685	1427140715	1783955745
17	9	4	0	713609325	2140667100	2854276425	3567885750
16	10	4	0	1213123626	3639150801	4852274427	6065398053
15	11	4	0	1764526140	5293321020	7057847160	8822373300
14	12	4	0	2205652956	6616662000	8822314956	11027967912
13	13	4	0	2375312100	7125636000	9500948100	11876260200
20	5	5	0	63103332	189274803	252378120	315481452
19	6	5	0	210334410	630926010	841260420	1051594830
18	7	5	0	570883170	1712537970	2283421140	2854304310
17	8	5	0	1284471045	3853226520	5137697565	6422168610
16	9	5	0	2426194485	7278345360	9704539845	12130734330
15	10	5	0	3881895744	11645368026	15527263740	19409159484
14	11	5	0	5293475460	15880066020	21173541480	26467016940
13	12	5	0	6175715364	18526749696	24702465060	30878180424
18	6	6	0	666038920	1997955960	2663994880	3330033160
17	7	6	0	1712623770	5137639650	6850263420	8562887190
16	8	6	0	3639297090	10917520185	14556817275	18196114365
15	9	6	0	6469809480	19408963860	25878773340	32348581560
14	10	6	0	9704693856	29113477536	38818171392	48522865248
13	11	6	0	12351394692	37053535428	49404930120	61756324812
12	12	6	0	13380683118	40141339056	53522022174	66902703702
16	7	7	0	4159167870	12477186150	16636354020	20795521890
15	8	7	0	8318301420	24954406620	33272708040	41591009460
14	9	7	0	13863778500	41590734900	55454513400	69318291900
13	10	7	0	19409265876	58227052884	77636318760	97045584636
12	11	7	0	22938201468	68813811612	91752013080	114690214548
14	8	8	0	15596775480	46789567110	62386342590	77983118070
13	9	8	0	24261567330	72783831120	97045398450	121306965780
12	10	8	0	31540028520	94619010486	126159039006	157699067526
11	11	8	0	34407266166	103220753454	137628019620	172035285786
12	9	9	0	35044420110	105132267240	140176687350	175221105360
11	10	9	0	42053285274	126158738706	168212023980	210265309254
10	10	10	0	46258637244	138774614388	185033251584	231291888828
27	1	1	1	206	606	812	1018
26	2	1	1	2765	8197	10962	13727
25	3	1	1	23790	71214	95004	118794
24	4	1	1	148603	445172	593775	742378
23	5	1	1	712764	2137356	2850120	3562884
22	6	1	1	2732002	8193458	10925460	13657462
21	7	1	1	8585148	25752012	34337160	42922308

20	8	1	1	22535513	67599532	90135045	112670558
19	9	1	1	50077170	150222930	200300100	250377270
18	10	1	1	95146051	285424139	380570190	475716241
17	11	1	1	155691666	467059554	622751220	778442886
16	12	1	1	220562979	661667916	882230895	1102793874
15	13	1	1	271460808	814361832	1085822640	1357283448
14	14	1	1	290851356	872530044	1163381400	1454232756
25	2	2	1	35789	106717	142506	178295
24	3	2	1	297193	890357	1187550	1484743
23	4	2	1	1782378	5342922	7125300	8907678
22	5	2	1	8195850	24580530	32776380	40972230
21	6	2	1	30049162	90130898	120180060	150229222
20	7	2	1	90141194	270398986	360540180	450681374
19	8	2	1	225348695	676001755	901350450	1126699145
18	9	2	1	475727395	1427123555	1902850950	2378578345
17	10	2	1	856304163	2568827547	3425131710	4281435873
16	11	2	1	1323371439	3970013931	5293385370	6616756809
15	12	2	1	1764491820	5293355340	7057847160	8822338980
14	13	2	1	2035949196	6107720604	8143669800	10179618996
23	3	3	1	2375568	7124832	9500400	11875968
22	4	3	1	13658658	40968642	54627300	68285958
21	5	3	1	60092604	180267516	240360120	300452724
20	6	3	1	210321826	630938594	841260420	1051582246
19	7	3	1	600908880	1802692320	2403601200	3004510080
18	8	3	1	1427155015	4281397835	5708552850	7135707865
17	9	3	1	2854295730	8562809970	11417105700	14271401430
16	10	3	1	4852300167	14556779523	19409079690	24261379857
15	11	3	1	7057878048	21173510592	28231388640	35289266688
14	12	3	1	8822349276	26466886524	35289235800	44111585076
13	13	3	1	9500984136	28502808264	38003792400	47504776536
21	4	4	1	75118758	225331392	300450150	375568908
20	5	4	1	315482310	946408320	1261890630	1577372940
19	6	4	1	1051597690	3154704410	4206302100	5257899790
18	7	4	1	2854307170	8562798530	11417105700	14271412870
17	8	4	1	6422175045	19266312780	25688487825	32110662870
16	9	4	1	12130740765	36391958460	48522699225	60653439990
15	10	4	1	19409169780	58227148980	77636318760	97045488540
14	11	4	1	26467027236	79400680164	105867707400	132334734636
13	12	4	1	30878192436	92634132864	123512325300	154390517736
19	5	5	1	1261903500	3785659020	5047562520	6309466020
18	6	5	1	3996019170	11987928810	15983947980	19979967150
17	7	5	1	10275433740	30826146780	41101580520	51377014260
16	8	5	1	21835287045	65505571560	87340858605	109176145650
15	9	5	1	38818236600	116454400920	155272637520	194090874120
14	10	5	1	58227354900	174681601380	232908956280	291136311180
13	11	5	1	74107503288	222322077432	296429580720	370537084008
12	12	5	1	80283146580	240848899200	321132045780	401415192360
17	6	6	1	11988031770	35963812170	47951843940	59939875710
16	7	6	1	29113711770	87340766370	116454478140	145568189910
15	8	6	1	58227389220	174681567060	232908956280	291136345500

14	9	6	1	97045574340	291136019460	388181593800	485227168140
13	10	6	1	135863780052	407590451268	543454231320	679318011372
12	11	6	1	160566257124	481697834436	642264091560	802830348684
15	7	7	1	66545519040	199636145280	266181664320	332727183360
14	8	7	1	124772839620	374317780980	499090620600	623863460220
13	9	7	1	194090977080	582272210520	776363187600	970454164680
12	10	7	1	252318294228	756953849652	1009272143880	1261590438108
11	11	7	1	275256255456	825767901504	1101024156960	1376280412416
13	8	8	1	218352424290	655056161760	873408586050	1091761010340
12	9	8	1	315397852770	946192327080	1261590179850	1576988032620
11	10	8	1	378477405306	1135430810514	1513908215820	1892385621126
11	9	9	1	420530330220	1261589909580	1682120239800	2102650570020
10	10	9	1	462583435314	1387748828466	1850332263780	2312915699094
24	2	2	2	446290	1335126	1781416	2227706
23	3	2	2	3564600	10686000	14250600	17815200
22	4	2	2	20491380	61450116	81941496	102432876
21	5	2	2	90145770	270394410	360540180	450685950
20	6	2	2	315496324	946396308	1261892632	1577388956
19	7	2	2	901386200	2704015600	3605401800	4506788000
18	8	2	2	2140768630	6422065650	8562834280	10703602910
17	9	2	2	4281495075	12844163475	17125658550	21407153625
16	10	2	2	7278517818	21835110726	29113628544	36392146362
15	11	2	2	10586899440	31760183520	42347082960	52933982400
14	12	2	2	13233615720	39700249992	52933865712	66167481432
13	13	2	2	14251572300	42754116300	57005688600	71257260900
22	3	3	2	27317160	81937440	109254600	136571760
21	4	3	2	150236658	450663642	600900300	751136958
20	5	3	2	630963762	1892817498	2523781260	3154745022
19	6	3	2	2103192520	6309411680	8412604200	10515796720
18	7	3	2	5708611480	17125599920	22834211400	28542822880
17	8	3	2	12844343655	38532631995	51376975650	64221319305
16	9	3	2	24261475095	72783923355	97045398450	121306873545
15	10	3	2	38818329264	116454308256	155272637520	194090966784
14	11	3	2	52934044176	158801370624	211735414800	264669458976
13	12	3	2	61756372860	185268277740	247024650600	308781023460
20	4	4	2	788717358	2366012220	3154729578	3943446936
19	5	4	2	3154784490	9464121810	12618906300	15773690790
18	6	4	2	9990091540	29969788420	39959879960	49949971500
17	7	4	2	25688661570	77065289730	102753951300	128442612870
16	8	4	2	54588323790	163763845245	218352169035	272940492825
15	9	4	2	97045745940	291135847860	388181593800	485227339740
14	10	4	2	145568564856	436703861880	582272426736	727840991592
13	11	4	2	185268974436	555804977364	741073951800	926342926236
12	12	4	2	200708076660	602122079832	802830156492	1003538233152
18	5	5	2	11988048930	35963795010	47951843940	59939892870
17	6	5	2	35964069570	107891462250	143855531820	179819601390
16	7	5	2	87341109570	262022324850	349363434420	436704543990
15	8	5	2	174682116180	524044752660	698726868840	873408985020
14	9	5	2	291136671540	873408109860	1164544781400	1455681452940
13	10	5	2	407591268084	1222771425876	1630362693960	2037953962044

12	11	5	2	481698699300	1445093575380	1926792274680	2408490973980
16	6	6	2	101898053400	305692650120	407590703520	509488756920
15	7	6	2	232909453920	698726371200	931635825120	1164545279040
14	8	6	2	436705080240	1310112151920	1746817232160	2183522312400
13	9	6	2	679318660020	2037952496580	2717271156600	3396589816620
12	10	6	2	883114240008	2649338347656	3532452587664	4415566827672
11	11	6	2	963397182384	2890187366976	3853584549360	4816981731744
14	7	7	2	499091255520	1497271226880	1996362482400	2495453737920
13	8	7	2	873409516980	2620224827220	3493634344200	4367043861180
12	9	7	2	1261591230900	3784769488500	5046360719400	6307951950300
11	10	7	2	1513909405008	4541723458272	6055632863280	7569542268288
12	8	8	2	1419290352480	4257865561950	5677155914430	7096446266910
11	9	8	2	1892386666170	5677154412930	7569541079100	9461927745270
10	10	8	2	2081625485940	6244869827196	8326495313136	10408120799076
10	9	9	2	2312916816210	6938744502690	9251661318900	11564578135110
21	3	3	3	200305368	600895152	801200520	1001505528
20	4	3	3	1051594830	3154707270	4206302100	5257896930
19	5	3	3	4206327840	12618880560	16825208400	21031536240
18	6	3	3	13320021420	39959805600	53279827020	66599847180
17	7	3	3	34251394320	102753874080	137005268400	171256662720
16	8	3	3	72784193625	218352001725	291136195350	363920388975
15	9	3	3	129394019880	388181439360	517575459240	646969476600
14	10	3	3	194091028560	582272159040	776363187600	970454216160
13	11	3	3	247024866816	741073735584	988098602400	1235123469216
12	12	3	3	267610309470	802829844180	1070440153650	1338050459970
19	4	4	3	5257939830	15773570670	21031510500	26289450330
18	5	4	3	19980027210	59939712690	79919739900	99899767110
17	6	4	3	59940004410	179819215290	239759219700	299699224110
16	7	4	3	145568361510	436704029190	582272390700	727840752210
15	8	4	3	291136620060	873408161340	1164544781400	1455681401460
14	9	4	3	485227494180	1455680474820	1940907969000	2426135463180
13	10	4	3	679318419780	2037952736820	2717271156600	3396589576380
12	11	4	3	802830781116	2408489676684	3211320457800	4014151238916
17	5	5	3	71927881740	215783181900	287711063640	359638945380
16	6	5	3	203795613450	611385733530	815181346980	1018976960430
15	7	5	3	465818221440	1397453428800	1863271650240	2329089871680
14	8	5	3	873409139460	2620225204740	3493634344200	4367043483660
13	9	5	3	1358636118840	4075906194360	5434542313200	6793178432040
12	10	5	3	1766227026564	5298677980596	7064905007160	8831132033724
11	11	5	3	1926792923328	5780376175392	7707169098720	9633962022048
15	6	6	3	543454798860	1630362127680	2173816926540	2717271721620
14	7	6	3	1164545484960	3493633640640	4658179125600	5822724610560
13	8	6	3	2037954418500	6113859051300	8151813469800	10189767888300
12	9	6	3	2943711592920	8831130087780	11774841680700	14718553267320
11	10	6	3	3532453836912	10597356177408	14129810014320	17662263851232
13	7	7	3	2329090283520	6987267967680	9316358251200	11645448534720
12	8	7	3	3784771770780	11354310387420	15139082158200	18923853928980
11	9	7	3	5046361800480	15139081077120	20185442877600	25231804678080
10	10	7	3	5550998268816	16652988896544	22203987165360	27754985434176
11	8	8	3	5677157475990	17031465761310	22708623237300	28385780713290

10	9	8	3	6938747745930	20816236210770	27754983956700	34693731702630
9	9	9	3	7709719119900	23129151945900	30838871065800	38548590177300
18	4	4	4	24975105870	74924584020	99899689890	124874795760
17	5	4	4	89909987310	269728842240	359638829550	449548816860
16	6	4	4	254744751690	764231977080	1018976728770	1273721480460
15	7	4	4	582273137160	1746816425640	2329089562800	2911362699960
14	8	4	4	1091761919820	3275281100520	4367043020340	5458804940160
13	9	4	4	1698295779180	5094882112320	6793177891500	8491473670680
12	10	4	4	2207784494916	6623346890160	8831131385076	11038915879992
11	11	4	4	2408491910916	7225469462484	9633961373400	12042453284316
16	5	5	4	305693400870	917078619600	1222772020470	1528465421340
15	6	5	4	815182144920	2445543243000	3260725387920	4075907532840
14	7	5	4	1746818175960	5240450512440	6987268688400	8734086864360
13	8	5	4	3056931537660	9170788667040	12227720204700	15284651742360
12	9	5	4	4415567296140	13246695221760	17662262517900	22077829814040
11	10	5	4	5298680647260	15896034374220	21194715021480	26493395668740
14	6	6	4	2037954959040	6113858630880	8151813589920	10189768548960
13	7	6	4	4075908596760	12227718342840	16303626939600	20379535536360
12	8	6	4	6623351274540	19870042712520	26493393987060	33116745261600
11	9	6	4	8831134051740	26493390984060	35324525035800	44155659087540
10	10	6	4	9714247799256	29142729992376	38856977791632	48571225590888
12	7	7	4	7569543301320	22708621015080	30278164316400	37847707617720
11	8	7	4	11354314591620	34062931882980	45417246474600	56771561066220
10	9	7	4	13877495131500	41632472781900	55509967913400	69387463044900
10	8	8	4	15612182766180	46836531451710	62448714217890	78060896984070
9	9	8	4	17346868689150	52040591202600	69387459891750	86734328580900
15	5	5	5	978218079720	2934652385820	3912870465480	4891088545200
14	6	5	5	2445545096280	7336631067480	9782176163760	12227721260040
13	7	5	5	4891089162960	14673263164560	19564352327520	24455441490480
12	8	5	5	7948019979900	23844052552320	31792072532220	39740092512120
11	9	5	5	10597359132360	31792070910600	42389430042960	52986789175320
10	10	5	5	11657095478064	34971277569246	46628373047220	58285468525284
13	6	6	5	5706271410840	17118806304600	22825077715440	28531349126280
12	7	6	5	10597359853080	31792070189880	42389430042960	52986789896040
11	8	6	5	15896039419260	47688105645180	63584145064440	79480184483700
10	9	6	5	19428492103020	58285462975740	77713955078760	97142447181780
11	7	7	5	18166900752000	54500693607360	72667594359360	90834495111360
10	8	7	5	24979489074540	74938453169580	99917942244120	124897431318660
9	9	7	5	27754986659400	83264949167400	111019935826800	138774922486200
9	8	8	5	31224361117950	93673066687200	124897427805150	156121788923100
12	6	6	6	12363587559480	37090747774080	49454335333560	61817922883560
11	7	6	6	21194718985440	63584141100480	84778860085920	105973579071360
10	8	6	6	29142738905280	87428194133280	116570933038560	145713671943840
9	9	6	6	32380819575300	97142438893500	129523258468800	161904078031500
10	7	7	6	33305985192480	99917937799680	133223922992160	166529908184640
9	8	7	6	41632481190300	124897422549900	166529903740200	208162384930500
8	8	8	6	46836542652900	140509599580350	187346142233250	234182684886150
9	7	7	7	47579976100800	142739913888000	190319889988800	237899866089600
8	8	7	7	53527474614900	160582401622500	214109876237400	267637350852300

Table 6 Icosahedral Edge Combinatorics of u irreducible representations (4 colors)

Partition	A <sub>u</sub>	T <sub>1u</sub>	G <sub>u</sub>	H <sub>u</sub>
30 0 0 0	0	0	0	0
29 1 0 0	0	1	1	1
28 2 0 0	3	11	14	17
27 3 0 0	32	105	137	164
26 4 0 0	221	689	910	1131
25 5 0 0	1166	3586	4749	5915
24 6 0 0	4912	14872	19784	24676
23 7 0 0	16874	50986	67860	84734
22 8 0 0	48620	146432	195052	243672
21 9 0 0	118996	357929	476925	595861
20 10 0 0	249995	751409	1001398	1251393
19 11 0 0	454727	1366183	1820910	2275637
18 12 0 0	720125	2162849	2882974	3602999
17 13 0 0	997248	2994747	3991995	4989243
16 14 0 0	1210944	3636264	4847208	6058152
15 15 0 0	1291834	3878798	5170622	6462330
28 1 1 0	6	23	29	35
27 2 1 0	97	309	406	503
26 3 1 0	897	2757	3654	4551
25 4 1 0	5902	17849	23751	29653
24 5 1 0	29588	89167	118755	148343
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11	8	6	5	15896034374220	47688110690220	63584145064440	79480179438660
10	9	6	5	19428486697620	58285468381140	77713955078760	97142441776380
11	7	7	5	18166896427680	54500697931680	72667594359360	90834490787040
10	8	7	5	24979483308780	74938458935340	99917942244120	124897425552900
9	9	7	5	27754981254000	83264954572800	111019935826800	138774917080800
9	8	8	5	31224354361200	93673073443950	124897427805150	156121782166350
12	6	6	6	12363582073980	37090752699000	49454334772980	61817916837540
11	7	6	6	21194712739200	63584147346720	84778860085920	105973572825120
10	8	6	6	29142730556940	87428201640780	116570932197720	145713662754660
9	9	6	6	32380811767500	97142446701300	129523258468800	161904070223700
10	7	7	6	33305977985280	99917945006880	133223922992160	166529900977440
9	8	7	6	41632472781900	124897430958300	166529903740200	208162376522100
8	8	8	6	46836532142400	140509609039800	187346141182200	234182673324600
9	7	7	7	47579968893600	142739921095200	190319889988800	237899858882400
8	8	7	7	53527465605900	160582410631500	214109876237400	267637341843300

